QFT

Notation

$d\Omega_k = \frac{d^3k}{(2\pi)^3}\frac{1}{2k_0}$
$xk = x_\mu k^\mu$
$k_0 = \sqrt{\vec{k}^2 + m^2}$

Math

$\epsilon^{ijk}\epsilon^{ijl} = 2\delta^{kl}$
$\epsilon^{kij}\epsilon^{klm} = \delta^{il}\delta^{jm} - \delta^{im}\delta^{jl}$

Noether

$\phi(x) \longrightarrow \phi'(x) = \phi(x) + \alpha\,\delta\phi(x)$
$\mathcal{L}[\phi', \partial\phi'](x) - \mathcal{L}[\phi, \partial\phi](x) = \alpha\partial_\mu J^\mu(x) + \alpha\Delta(x)$
$\partial_\mu j^\mu = \Delta - \frac{\delta S}{\delta\phi}\delta\phi$ where $j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi - J^\mu$

Lorentz

$$\Lambda^\mu{}_\nu(\vec{\eta}, \vec{\theta}) \equiv \exp\begin{pmatrix} 0 & \eta^1 & \eta^2 & \eta^3 \\ \eta^1 & 0 & -\theta^3 & \theta^2 \\ \eta^2 & \theta^3 & 0 & -\theta^1 \\ \eta^3 & -\theta^2 & \theta^1 & 0 \end{pmatrix} \text{ spans } SO(1,3)$$

Klein-Gordon

$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m}{2}\phi^2$
$\pi = \partial_0\phi \Rightarrow H = \frac{1}{2}\pi^2 + \partial_i\phi\partial_i\phi + \frac{m}{2}\phi^2 \Rightarrow [\phi(\vec{x}, t), \pi(\vec{y}, t)] = i\delta^3(\vec{x} - \vec{y})$
$\phi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\vec{x}}\phi(\vec{k}, t)$
$a(\vec{k}) = k_0\phi(\vec{k}) + i\pi(\vec{k}) \Rightarrow [a(\vec{k}), a^\dagger(\vec{p})] = (2\pi)^3 2k_0\delta^3(\vec{k} - \vec{p})$
$\phi(x^\mu) = \int \frac{d^3k}{(2\pi)^3 2k_0}\left(a(\vec{k})e^{-ik_\mu x^\mu} + a^\dagger(\vec{k})e^{ik_\mu x^\mu}\right)$

Spinors

$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
$\sigma^{\mu\dagger} = \sigma^\mu \quad (\sigma^\mu)^2 = 1$
$[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$
$\{\sigma^i, \sigma^j\} = 2\delta^{ij}$
$\sigma_\mu\bar{\sigma}_\nu + \sigma_\nu\bar{\sigma}_\mu = 2\eta_{\mu\nu} \quad \Rightarrow X_\mu\sigma^\mu X_\nu\bar{\sigma}^\nu = X_\mu X^\mu = \det(X_\mu\sigma^\mu)$
$\epsilon\sigma^\mu\epsilon^T = \bar{\sigma}^{\mu *}$
$\Lambda_L \equiv \exp(-\frac{1}{2}(\eta^i + i\theta^i)\sigma^i)$ spans $SL(2, \mathbb{C})$
$\Lambda_R \equiv \exp(-\frac{1}{2}(-\eta^i + i\theta^i)\sigma^i)$
$\epsilon\Lambda_L\epsilon^T = \Lambda_R^* \quad \Leftrightarrow \quad \epsilon\Lambda_R\epsilon^T = \Lambda_L^*$
$\Lambda_L^\dagger\bar{\sigma}_\mu\Lambda_L = \Lambda_\mu{}^\nu\bar{\sigma}_\nu \quad \Leftrightarrow \quad \Lambda_R^\dagger\sigma_\mu\Lambda_R = \Lambda_\mu{}^\nu\sigma_\nu$
$\Leftrightarrow \Lambda_L\sigma^\mu\Lambda_L^\dagger = \Lambda_\nu{}^\mu\sigma^\nu \quad \Leftrightarrow \quad \Lambda_R\bar{\sigma}^\mu\Lambda_R^\dagger = \Lambda_\nu{}^\mu\bar{\sigma}^\nu$
$\exp(A \otimes 1 + 1 \otimes B) = \exp(A) \otimes \exp(B)$
$(1/2, 0) \rightarrow \Lambda_L$ truly speaking it's not a representation of the proper othochronous Lorentz group because $\Lambda_L(\text{half-turn})\Lambda_L(\text{half-turn}) = -1 \neq \Lambda_L(\text{full-turn})$. It is a representation of its covering group : $SL(2, \mathbb{C})$
$(0, 1/2) \rightarrow \Lambda_R$
$(1/2, 0) + (0, 1/2) \rightarrow \begin{pmatrix} \Lambda_L & 0 \\ 0 & \Lambda_R \end{pmatrix}$
$(1/2, 1/2) \rightarrow \Lambda_L \otimes \Lambda_R$
$(\epsilon \otimes \epsilon)(\Lambda_L \otimes \Lambda_L^*)(\epsilon \otimes \epsilon)^{-1} = (\Lambda_R^* \otimes \Lambda_R)$
$\text{Partity}(\Lambda_L \otimes \Lambda_L^*)\text{Partity} = (\Lambda_R \otimes \Lambda_R^*)$

Dirac

$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad \gamma^0\gamma^\mu\gamma^0 = \gamma_\mu$
$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$
$\bar{\psi} = \psi^\dagger\gamma^0$

Vecteur

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ & 0 & -B^3 & B^2 \\ & & 0 & -B^1 \\ & & & 0 \end{pmatrix} \quad \begin{matrix} A^\mu = (\phi\ \vec{A}) \\ E^i = -\partial_i\phi - \partial_0 A^i \\ B^i = \epsilon^{ijk}\partial_j A^k \end{matrix}$$

Gupta-Bleuler

$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu$
$T^{\mu\nu} = -\partial^\mu A^\rho\partial^\nu A_\rho - \eta^{\mu\nu}\mathcal{L}$
$\pi^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_0 A_\mu)} = -\partial_0 A^\mu \Rightarrow \mathcal{H} = -\frac{1}{2}\pi_\mu\pi^\mu - \frac{1}{2}\partial_i A_\mu\partial_i A^\mu$
$\Rightarrow [A_\mu(\vec{x}, t), \pi^\nu(\vec{y}, t)] = i\delta_\mu^\nu\delta^3(\vec{x} - \vec{y})$
$A_\mu(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} A_\mu(\vec{k})$ idem for π^μ
$a_\mu(\vec{k}) = k_0 A_\mu(\vec{k}) - i\pi_\mu(\vec{k}) \Rightarrow [a_\mu(\vec{k}), a_\nu^\dagger(\vec{p})] = -\eta_{\mu\nu}2k_0(2\pi)^3\delta^3(\vec{k} - \vec{p})$
$H = \int d^3x\mathcal{H} = -\frac{1}{2}\eta^{\mu\nu}\int \frac{d^3k}{(2\pi)^3} a_\mu^\dagger(\vec{k})a_\nu(\vec{k}) = -\int d\Omega_k k_0 a_\mu^\dagger(\vec{k})a^\mu(\vec{k})$
$P^\mu = -\int d\Omega_k k^\mu a_\nu^\dagger(\vec{k})a^\nu(\vec{k})$
$e^{iHt}a_\mu(\vec{k})e^{-iHt} = a_\mu(\vec{k})e^{-ik_0 t}$ and $a^\dagger$ gets $+ik_0 t$
$A_\mu(x) = \int d\Omega_k\left(a_\mu(\vec{k})e^{-ikx} + a_\mu^\dagger(\vec{k})e^{ikx}\right)$

Cosmology

General Relativity

Units	$M_P = \sqrt{\frac{\hbar c}{G_N}} = 1.2209 \cdot 10^{19}\text{GeV}$	1
Metric	$g_{\mu\nu}$	0
Christoffel	$\Gamma^\mu_{\rho\sigma} = \frac{1}{2}g^{\mu\alpha}(g_{\alpha\sigma,\rho} + g_{\alpha\rho,\sigma} - g_{\sigma\rho,\alpha})$	1
Riemann	$R^\mu{}_{\nu\rho\sigma} = \Gamma^\mu_{\nu\rho,\sigma} - \Gamma^\mu_{\nu\sigma,\rho} + \Gamma^\alpha_{\nu\rho}\Gamma^\mu_{\alpha\sigma} - \Gamma^\alpha_{\nu\sigma}\Gamma^\mu_{\alpha\rho}$	2
Ricci	$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$ and $R = R^\mu{}_\mu = R^{\alpha\beta}{}_{\alpha\beta}$	2
Einstein eq	$G_{\mu\nu} - \lambda g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \lambda g_{\mu\nu} = \frac{8\pi}{M_P^2}T_{\mu\nu}$	2
FLRW	$g = dt^2 - a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2(\sin^2(\theta)d\phi^2 + d\theta^2)\right)$	-2
$\Rightarrow$	$G_{\mu\nu} = g_{\mu\nu}\text{diag}(3(\tilde{G} - 2\frac{\ddot{a}}{a}), \tilde{G}, \tilde{G}, \tilde{G}) \mid \tilde{G} = 2\frac{\ddot{a}}{a} + (\frac{\dot{a}}{a})^2 + \frac{k}{a^2}$	2
EM fluid	$T_{\mu\nu} = (p + \rho)u_\mu u_\nu - pg_{\mu\nu}$	4
Friedmann	$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \frac{\lambda}{3} = \frac{8\pi}{3M_P^2}\rho \quad (\star)$	2
	$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} - \lambda = -\frac{8\pi}{M_P^2}p \quad (\star\star)$	2

Cosmology

- ⋆ $\hbar = 6.582 \cdot 10^{-16}\text{eV s} = 2.0872 \cdot 10^{-23}\text{eV yr} \Rightarrow 1\text{pc} = 1.5619 \cdot 10^{23}\text{eV}^{-1}$
- ⋆ Redshift: $z = \frac{\lambda_0 - \lambda_1}{\lambda_1} = a_0/a_1 - 1$
- ⋆ Critical density: $\rho_c = \frac{3H^2}{8\pi G} = \frac{3H^2M_P^2}{8\pi}$
- ⋆ Abondances: $\Omega_m = \frac{\rho_m}{\rho_c} \quad \Omega_\gamma = \frac{\rho_\gamma}{\rho_c} \quad \Omega_\Lambda = \frac{\lambda}{3H^2} \quad \Omega_k = -\frac{k}{\dot{a}^2}$
- ⋆ Friedmann: $1 = \Omega_\Lambda + \Omega_m + \Omega_\gamma + \Omega_k$
  $\frac{3a^3}{8\pi G}((\dot{\star}) - \frac{\dot{a}}{a}(\star\star) + 3\frac{\dot{a}}{a}(\star)) = (\rho a^3)^\cdot + p(a^3)^\cdot = 0$
- ⋆ $1 = \frac{H_0^2}{H^2}\left\{\Omega_{\Lambda 0} + \Omega_{m0}(\frac{a_0}{a})^3 + \Omega_{\gamma 0}(\frac{a_0}{a})^4 + \Omega_{k0}(\frac{a_0}{a})^2\right\}$
- ⋆ Look back time: $t_l = \int_{t_1}^{t_0} dt = \int_{a_1}^{a_0}\frac{da}{aH}$
- ⋆ Light path: $ds^2 = 0 \quad \Rightarrow r_1 = \frac{1}{\sqrt{k}}\sin\left(\sqrt{k}\int_{t_1}^{t_0}\frac{dt}{a}\right)$
- ⋆ Luminosity dist.: (flux through eye) = (flux at emission)(eye surface)$\frac{1}{4\pi d_L^2}$
  $\Rightarrow d_L = a_0(z+1)r \Rightarrow d_L(z) = a_0(z+1)\frac{1}{\sqrt{k}}\sin(\sqrt{k}\int_{a_1}^{a_0}\frac{da}{a^2H})$
- ⋆ Ω_γ domination: $a \propto \sqrt{t} \Rightarrow H = \frac{1}{2t}$
- ⋆ Ω_m domination: $a \propto t^{2/3} \Rightarrow H = \frac{2}{3t}$
- ⋆ Ω_Λ domination: $H = \sqrt{\frac{\lambda}{3}} \Rightarrow a \propto \exp(\sqrt{\frac{\lambda}{3}}t)$
- ⋆ Hubble: $l(t) = \text{dist}((t, 0), (t, r)) \Rightarrow \dot{l} = Hl \Rightarrow l(t) = \frac{a(t)}{a(t_0)}l(t_0)$
  $l(t) = \text{dist}((t, 0), (t, r(t))_\gamma) \Rightarrow \dot{l} = Hl \pm 1 \Rightarrow l(t) = \pm\int_{t_0}^t \frac{a(t)}{a(t')}dt'$

Statistical physics

- ⋆ Partition f. (g spin, k mode): $Z_{B,F} = \exp\left(\mp g\sum_k \ln(1 \mp e^{-\frac{E(k)-\mu}{T}})\right)$
- ⋆ Mode occupation: $n_{B,F}(k) = \frac{g}{\exp(\frac{E(k)-\mu}{T}) \mp 1}$
- ⋆ $E = -\frac{\partial}{\partial\beta}\ln Z$ and $F = -T\ln Z$ and $N = T\frac{\partial}{\partial\mu}\ln Z$
- ⋆ Number density : $n = \int \frac{d^3p}{(2\pi)^3}n_{B,F}(\vec{p})$
- ⋆ $\mu = 0$, $E(\vec{k}) = |\vec{k}|$ and $\vec{k} \in \frac{2\pi}{\sqrt[3]{V}}\mathbb{Z}^3 \Rightarrow \sum_k \rightarrow V\int\frac{d^3k}{(2\pi)^3}$:
  Energy density : $\rho = \int \frac{d^3p}{(2\pi)^3}|\vec{p}|n_{B,F}(\vec{p}) = \frac{\pi^2}{30}gT^4 \cdot \begin{cases} 1 & \text{B} \\ \frac{7}{8} & \text{F} \end{cases}$
  Total effective nbr of massless degrees of freedom: $g_* = \sum_B g_i + \frac{7}{8}\sum_F g_i$
  Number density : $n = \frac{\zeta(3)}{\pi^2}gT^3 \cdot \begin{cases} 1 & \text{B} \\ \frac{3}{4} & \text{F} \end{cases}$
  Entropy: $S = \ln Z + E/T = \frac{2\pi^2}{45}gT^3 \cdot \begin{cases} 1 & \text{B} \\ \frac{7}{8} & \text{F} \end{cases}$
  Pressure: $p = -\left.\frac{\partial F}{\partial V}\right|_T = \frac{\pi^2}{90}gT^4 \cdot \begin{cases} 1 & \text{B} \\ \frac{7}{8} & \text{F} \end{cases} \Rightarrow p = \rho/3$
  Time (radiation domain): $t = \frac{M_0}{T^2} = \sqrt{\frac{3 \cdot 30}{32\pi^2 g_*}}\frac{M_p}{T^2} = \frac{2.40828\text{second}}{\sqrt{g_*}(T/\text{MeV})^2}$
- ⋆ $E(\vec{k}) = m + \frac{k^2}{2m}$ and $e^{\frac{E-\mu}{T}} \mp 1 \approx e^{\frac{E-\mu}{T}}$
  Number density: $n \approx g\left(\frac{mT}{2\pi}\right)^{3/2}e^{-\frac{m-\mu}{T}}$